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On certain geometric and homotopy properties of closed symplectic manifolds

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Abstract

It is well known that closed Kähler manifolds have certain homotopy properties which do not hold for symplectic manifolds. Here we survey interconnections between those properties.

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1. Introduction

Homotopy properties of closed symplectic manifolds attract the attention of geometers since the classical papers of Sullivan [26] and Thurston [27]. On one hand, "soft" homotopy techniques help in the solution of many "hard" problems in symplectic geometry, cf. [3,7, 20,24,29]. On the other hand, it is still unknown if there are specific homotopy properties of closed manifolds dependent on the existence of symplectic structures on them. It turns out that symplectic manifolds violate many specific homotopy conditions shared by the

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Kähler manifolds (which form a subclass of symplectic manifolds). In particular, if M is a closed Kähler manifold then the following holds:

(1) all the odd-degree Betti numbers $b_{2i+1}(M)$ are even;
(2) M has the Hard Lefschetz property;
(3) all Massey products (of all orders) in M vanish.

It is well known (and we shall see it below) that closed symplectic manifolds violate *all* the homotopy properties (1)–(3). However, it is not clear whether properties (1)–(3) are *independent* or not, in case of closed symplectic manifolds or certain classes of these ones. In other words, can a combination of the type

(1)–(2)–non-(3)

be realized by a closed symplectic manifold (possibly, with prescribed properties). The knowledge of an answer to this question might shed a new light on the whole understanding of closed symplectic manifolds.

In Theorem 3.1 we have summarized our knowledge by writing down the corresponding tables. We have considered two classes of symplectic manifolds: the class of symplectically aspherical symplectic manifolds and the class of simply-connected symplectic manifolds. Recall that a symplectically aspherical manifold is a symplectic manifold (M,ω) such that $\omega|\pi_2(M)=0$, i.e.,

$$\int_{S^2} f^{\#}\omega = 0$$

for every map $f: S^2 \to M$. In view of the Hurewicz Theorem, a closed symplectically aspherical manifold always has a non-trivial fundamental group. It is well known that symplectically aspherical manifolds play an important role in geometry and topology of symplectic manifolds [6,8,11,23,25].

The next topic of the paper is about symplectically harmonic forms on closed symplectic manifolds. Brylinski [2] and Libermann (Thesis, see [15]) have introduced the concept of a *symplectic star operator* $*$ on a symplectic manifold. In a sense, it is a symplectic analog of the Hodge star operator which is defined in terms of the given symplectic structure ω. Using this operator, one defines a symplectic codifferential $\delta := (-1)^{k+1}(*d*)$, $\deg\delta = -1$. Now we define *symplectically harmonic* differential forms α by the condition

$$\delta\alpha = 0, \qquad d\alpha = 0.$$

Let $\Omega^*_{\mathrm{hr}}(M,\omega)$ denote the space of all symplectically harmonic forms on M. Clearly, the space $H^k_{\mathrm{hr}}(M) := \Omega^k_{\mathrm{hr}}/(\Omega^k_{\mathrm{hr}} \cap \operatorname{Im} d)$ is a subspace of the de Rham cohomology space $H^k(M)$.

Here we also have an interesting relation between geometry and homotopy theory. For example, Mathieu [18] proved that $H^k_{\mathrm{hr}}(M,\omega) = H^k(M)$ if and only if M has the Hard Lefschetz property. We will also see that the Lefschetz map

$$L^k : H^{m-k}(M) \to H^{m+k}(M), \quad \dim M = 2m$$

(multiplication by $[\omega]^k$) plays an important role in studying of $H^k_{\mathrm{hr}}(M,\omega)$.

We set $h_k(M,\omega) = \dim H^k_{\mathrm{hr}}(M,\omega)$. According to Yan [31], the following question was posed by Boris Khesin and Dusa McDuff.

Question. Are there closed manifolds endowed with a continuous family ω_t of symplectic structures such that $h_k(M,\omega_t)$ varies with respect to t?

Yan [31] constructed a closed 4-dimensional manifold M with varying $h_3(M)$. So, he answered affirmatively the above question.

(Actually, Proposition 4.1 from [31] is wrong, the Kodaira–Thurston manifold is a counterexample, but its Corollary 4.2 from [31] is correct because it follows from our Lemma 4.4. Hence, the whole construction holds.)

However, the Yan's proof was essentially 4-dimensional. Indeed, Yan [31] wrote:

"For higher-dimensional closed symplectic manifolds, it is not clear how to answer the question in the beginning of this section", i.e., the above stated question.

In this note we prove the following result (Theorem 4.6): There exists at least one 6-dimensional indecomposable closed symplectic manifold N with varying $h_5(N)$.

Moreover, Yan remarked that there is no 4-dimensional closed symplectic *nilmanifolds* M with varying $\dim H^*_{\mathrm{hr}}(M)$. On the contrary, our example is a certain 6-dimensional nilmanifold.

2. Preliminaries and notation

Given a topological space X, let $(\mathcal{C}M_X, d)$ be the Sullivan model of X, that is, a certain natural commutative DGA algebra over the field of rational numbers $\mathbb{Q}$ which is a homotopy invariant of X, see [4,29,26] for details. Furthermore, if X is a nilpotent CW-space of finite type then $(\mathcal{C}M_X, d)$ completely determines the rational homotopy type of X.

A space X is called *formal* if there exists a DGA-morphism

$$\rho : (\mathcal{C}M_X, d) \to \bigl(H^*(X;\mathbb{Q}), 0\bigr)$$

inducing isomorphism on the cohomology level. Recall that every closed Kähler manifold in formal [4].

We refer the reader to [14,19,24] for the definition of Massey products. It is well known and easy to see that Massey products yield an obstruction to formality [4,24,29]. In other words, if the space is formal then all Massey products must be trivial. Thus, all the Massey products in every Kähler manifold vanish.

We need also the following result of Miller [22]:

Theorem 2.1. *Every closed simply-connected manifold M of dimension $\leqslant 6$ is formal. In particular, all Massey products in M vanish.*

The next homotopy property related to symplectic (in particular, Kähler) structures is the *Hard Lefschetz property*. Given a symplectic manifold (M^{2m},ω), we denote by $[\omega] \in H^2(M)$ the de Rham cohomology class of ω. Furthermore, we denote by $L_\omega : \Omega^k(M) \to \Omega^{k+2}(M)$ the multiplication by ω and by $L_{[\omega]} : H^k(M) \to H^{k+2}(M)$ the induced homomorphism in the de Rham cohomology $H^*(M)$. As usual we write L instead of L_ω or $L_{[\omega]}$ if there is no danger of confusion. We say that a symplectic manifold (M^{2m}, ω) has the *Hard Lefschetz property* if, for every k, the homomorphism

$$L^k : H^{m-k}(M) \to H^{m+k}(M)$$

is surjective. In view of the Poincaré duality, for closed manifolds M it means that every L^k is an isomorphism. We need also the following result of Gompf [7, Theorem 7.1].

Theorem 2.2. *For any even dimension $n \geqslant 6$, finitely presented group G and integer b there is a closed symplectic n-manifold M with $\pi_1(M) \cong G$ and $b_i(M) \geqslant b$ for $2 \leqslant i \leqslant n-2$, such that M does not satisfy the Hard Lefschetz condition. Furthermore, if $b_1(G)$ is even then all degree-odd Betti numbers of M are even.*

We denote such manifold M by $M(n, G, b)$.

Remark 2.3. Theorem 7.1 in [7] is formulated in a slightly different way, but the *proof* is based on constructing of M by some "symplectic summation" in a way to violate the Hard Lefschetz property.

In our explicit constructions we will need some particular classes of manifolds, namely, *nilmanifolds*, respectively *solvmanifolds*. These are homogeneous spaces of the form G/Γ, were G is a simply connected nilpotent, respectively solvable Lie group and Γ is a co-compact discrete subgroup (i.e., a *lattice*). The most important information for us is the following (see, e.g., [29] for the proofs):

Recollection 2.4.

(i) Let $\mathfrak{g}$ be a nilpotent Lie algebra with structural constants c_k^{ij} with respect to some basis, and let $\{\alpha_1, \dots, \alpha_n\}$ be the dual basis of $\mathfrak{g}^*$. Then the differential in the Chevalley–Eilenberg complex $(\Lambda^*\mathfrak{g}^*, d)$ is given by the formula

$$d\alpha_k = -\sum_{1 \leqslant i < j < k} c_k^{ij} \alpha_i \wedge \alpha_j.$$

(ii) Let $\mathfrak{g}$ be the Lie algebra of a simply connected nilpotent Lie group G. Then, by Malcev's theorem [17], G admits a lattice if and only if $\mathfrak{g}$ admits a basis such that all the structural constants are rational. Moreover, this lattice is unique up to an automorphism of G.

(iii) Let G and $\mathfrak{g}$ be as in (ii), and suppose that G admits a lattice Γ. By Nomizu's theorem, the Chevalley–Eilenberg complex $(\Lambda^*\mathfrak{g}^*, d)$ is quasi-isomorphic to the de Rham complex of G/Γ. Moreover, $(\Lambda^*\mathfrak{g}^*, d)$ is a minimal differential algebra, and hence it is isomorphic to the minimal model of G/Γ:

$$\left(\Lambda^*\mathfrak{g}^*, d\right) \cong (\mathcal{C}M_{G/\Gamma}, d).$$

Also, any cohomology class $[a] \in H^k(G/\Gamma)$ contains a homogeneous representative α. Here we call the form α *homogeneous* if the pullback of α to G is left invariant.

Let ω_0 be the standard symplectic form on $\mathbb{CP}^m$. Recall that every closed symplectic manifold (M^{2n}, ω) with integral form ω can be symplectically embedded into $\mathbb{CP}^m$ for m large enough, with the (known) smallest possible value of m equal to $n(n+1)$ [10,28]. We will use the *blow-up* construction with respect to such embedding [20,25]. We need the following results.

Theorem 2.5. *Let (M^{2n}, ω) be a closed connected symplectic manifold, let $i:(M, \omega) \to (\mathbb{CP}^m, \omega_0)$ be a symplectic embedding, and let $\widetilde{\mathbb{CP}^m}$ be the blow-up along i. Then the following holds*:

- (i) *$\widetilde{\mathbb{CP}^m}$ is a simply-connected symplectic manifold*;
- (ii) *if there exists i such that $b_{2i+1}(M)$ is odd, then there exists k such that $b_{2k+1}(\widetilde{\mathbb{CP}^m})$ is odd*;
- (iii) *if M possesses a non-trivial Massey triple product and $m-n \geqslant 4$, then $\widetilde{\mathbb{CP}^m}$ possesses a non-trivial Massey triple product. Moreover, if there is a non-trivial Massey product $\langle \alpha, \beta, [\omega] \rangle \in H^*(M)$, $\alpha, \beta \in H^*(M)$, then $\widetilde{\mathbb{CP}^m}$ possesses a non-trivial Massey triple product even for $m-n=3$.*

Proof. (i) and (ii) are proved in [20], (i) and (iii) are proved in [25], cf. also [1]. □

3. Relation between homotopy properties of closed symplectic manifolds

Theorem 3.1. *The relations between the Hard Lefschetz property, evenness of odd-degree Betti numbers and vanishing of the Massey products for closed symplectic manifolds are given by the following tables*:

Table 1: *Symplectically aspherical case*;
Table 2: *Simply-connected case.*

Table 1
Symplectically aspherical symplectic manifolds

Triviality of Massey products	Hard Lefschetz property	Evenness of b_{2i+1}	
yes	yes	yes	Kähler ($\mathbb{T}^{2n}$)
yes	yes	no	Impossible
yes	no	yes	?
yes	no	no	?
no	yes	yes	?
no	yes	no	Impossible
no	no	yes	$K \times K$
no	no	no	K

Table 2
Simply-connected symplectic manifolds

Triviality of Massey products	Hard Lefschetz property	Evenness of b_{2i+1}	
yes	yes	yes	Kähler ($\mathbb{CP}^n$)
yes	yes	no	Impossible
yes	no	yes	M(6,{e},0)
yes	no	no	?
no	yes	yes	?
no	yes	no	Impossible
no	no	yes	$\widetilde{\mathbb{CP}^5} \times \widetilde{\mathbb{CP}^5}$
no	no	no	$\widetilde{\mathbb{CP}^5}$

The word *Impossible* in the table means that there is no closed symplectic manifold (aspherical or simply connected) that realizes the combination in the corresponding line.

The sign **?** means that we (the authors) do not know whether a manifold with corresponding properties exists.

Proof. We prove the theorem via line-by-line analysis of Tables 1 and 2. $\square$

Line 1 in Tables 1 and 2. For closed Kähler manifolds, the Hard Lefschetz property is proved in [9], the evenness of b_{2i+1} follows from the Hodge theory [30], the triviality of Massey products follows from the formality of any closed Kähler manifold [4].

One can ask if there are non-Kähler manifolds having the properties from line 1. In the symplectically aspherical case the answer is affirmative. Let $G = \mathbb{R} \times_\phi \mathbb{R}^2$ be the semi-direct product determined by the one-parameter subgroup $\phi(t) = \mathrm{diag}(e^{kt}, e^{-kt}),\ t \in \mathbb{R}$, $e^k + e^{-k} \neq 2$. One can check that G contains a lattice, say Γ. Then the compact solvmanifold

$$M = G/\Gamma \times S^1$$

is symplectic and has the same minimal model as the Kähler manifold $S^2 \times T^2$. Hence such manifold fits into line 1. It cannot be Kähler, since it admits no complex structure. The latter follows from the Kodaira–Yau classification of compact complex surfaces (see [29] for details).

Line 2 in Tables 1 and 2. Any manifold satisfying the Hard Lefschetz property must have *even* b_{2i+1}. Indeed, consider the usual non-singular pairing $p : H^{2k+1}(M) \otimes H^{2m-2k-1}(M) \to \mathbb{R}$ of the form

$$p\big([\alpha], [\beta]\big) = \int_M \alpha \wedge \beta.$$

Define a skew-symmetric bilinear form $\langle -, - \rangle : H^{2k+1}(M) \otimes H^{2k+1}(M) \to \mathbb{R}$ via the formula

$$\big\langle [\alpha], [\gamma] \big\rangle = p\big([\alpha], L^{m-2k-1}[\gamma]\big),$$

for $[\alpha], [\gamma] \in H^{2k+1}(M)$. Since this form is non-degenerate and skew-symmetric, its domain $H^{2k+1}(M)$ must be even-dimensional, i.e., b_{2k+1} is even.

Line 3 in Table 1. We do not know any non-simply-connected (and, in particular, symplectically aspherical) examples to fill in this line.

Line 3 in Table 2. We use Theorem 2.2 with $n = 6$ and $G = \{e\}$. Then, for every b, the corresponding manifold $M(6, \{e\}, b)$ has even odd-degree Betti numbers and does not have the Hard Lefschetz property. Furthermore, all the Massey products in M vanish by 2.1.

Line 4 and 5 in Tables 1 and 2. We do not know any examples to fill in these lines.

Line 6 in Tables 1 and 2. This is impossible, see the argument concerning line 2.

Lines 7 and 8 in Table 1. Consider the Kodaira–Thurston manifold K [27]. Recall that this manifold is defined as a nilmanifold

$$K = N_3/\Gamma \times S^1,$$

where N_3 denotes the 3-dimensional nilpotent Lie group of triangular unipotent matrices and Γ denotes the lattice of such matrices with integer entries. One can check that the Chevalley–Eilenberg complex of the Lie algebra $\mathfrak{n}_3$ is of the form

$$\big(\Lambda(e_1, e_2, e_3), d\big), \qquad de_1 = de_2 = 0, \qquad de_3 = e_1 e_2$$

with $|e_i| = 1$. We have already mentioned that the minimal model of any nilmanifold N/Γ is isomorphic to the Chevalley–Eilenberg complex of the Lie algebra $\mathfrak{n}$. In particular, one can get the minimal model of the Kodaira–Thurston manifold in the form

$$\big(\Lambda(x, e_1, e_2, e_3), d\big), \qquad dx = de_1 = de_2 = 0, \qquad de_3 = e_1 e_2$$

with degrees of all generators equal 1. One can check that the vector space $H^1(K)$ has the basis $\{[x], [e_1], [e_2]\}$. Hence, $b_1(K) = 3$, which also shows that K does not have the Hard Lefschetz property. Furthermore, K possesses a symplectic form ω with $[\omega] = [e_1 e_3 + e_2 x]$, and one can prove that the Massey triple product $\langle [e_1], [e_1], [\omega] \rangle$ is non-trivial. Thus, K realizes line 8 of Table 1.

Finally, $K \times K$ realizes line 7 of Table 1.

Lines 7 and 8 in Table 2. We use Theorem 2.5. Consider a symplectic embedding $i : K \to \mathbb{CP}^m$, $m \geqslant 5$, and perform the blow-up along i. Then, by 2.5(i), $\widetilde{\mathbb{CP}^m}$ is simply-connected. Furthermore, it realizes line 8 of Table 2 by 2.5(ii) and 2.5(iii).

Finally, $\widetilde{\mathbb{CP}^m} \times \widetilde{\mathbb{CP}^m}$ realizes line 7 of Table 2. □

Remark 3.2. The result of Lupton [16] shows that the problem of constructing of a non-formal manifold with the Hard Lefschetz property turns our to be very delicate. In [16] there is an example of a DGA, whose cohomology has the Hard Lefschetz property, but which is not *intrinsically* formal. This means that there is also a *non-formal* minimal algebra with the same cohomology ring. Sometimes, using Browder–Novikov theory, one can construct a smooth closed manifold M with such non-formal Sullivan minimal model. However, there is no way in sight to get a symplectic structure on M.

4. Flexible symplectic manifolds

Let (M^{2m}, ω) be a symplectic manifold. It is known that there exists a unique non-degenerate Poisson structure Π associated with the symplectic structure (see, for example, [15,29]). Recall that Π is a skew symmetric tensor field of order 2 such that $[\Pi, \Pi] = 0$, where $[-, -]$ is the Schouten–Nijenhuis bracket.

The Koszul differential $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$ is defined for Poisson, in particular symplectic, manifolds as

$$\delta = \bigl[i(\Pi), d\bigr].$$

Brylinski has proved in [2] that the Koszul differential is a symplectic codifferential of the exterior differential with respect to the symplectic star operator. We choose the volume form associated to the symplectic form, say $v_M = \omega^m/m!$. Then we define the symplectic star operator

$$* : \Omega^k(M) \to \Omega^{2m-k}(M)$$

by the condition $\beta \wedge (*\alpha) = \Lambda^k(\Pi)(\beta, \alpha)v_M$, for all $\alpha, \beta \in \Omega^k(M)$. It turns out to be that

$$\delta = (-1)^{k+1}(* \circ d \circ *).$$

Definition 4.1. A k-form α on the symplectic manifold M is called *symplectically harmonic*, if $d\alpha = 0 = \delta\alpha$.

We denote by $\Omega^k_{\mathrm{hr}}(M)$ the space of symplectically harmonic k-forms on M. We set

$$H^k_{\mathrm{hr}}(M, \omega) = \Omega^k_{\mathrm{hr}}(M)/\bigl(\mathrm{Im}\, d \cap \Omega^k_{\mathrm{hr}}(M)\bigr),$$
$$h_k(M) = h_k(M, \omega) = \dim\, H^k_{\mathrm{hr}}(M, \omega).$$

We say that a de Rham cohomology class is *symplectically harmonic* if it contains a symplectically harmonic representative, i.e., if it belongs to the subgroup $H^*_{\mathrm{hr}}(M)$ of $H^*(M)$.

Definition 4.2. We say that a closed smooth manifold M is *flexible*, if M possesses a continuous family of symplectic forms ω_t, $t \in [a, b]$, such that $h_k(M, \omega_a) \neq h_k(M, \omega_b)$ for some k.

So, the McDuff–Khesin Question (see the introduction) asks about existence of flexible manifolds.

In order to prove our result on the existence of flexible 6-dimensional nilmanifolds, we need some preliminaries. The following lemma is proved in [12] and generalizes an observation of Yan [31].

Lemma 4.3. *For any symplectic manifold (M^{2m}, ω) and $k = 0, 1, 2$ we have*

$$H_{\mathrm{hr}}^{2m-k}(M) = \mathrm{Im}\{L^{m-k} : H^k(M) \to H^{2m-k}(M)\} \subset H^{2m-k}(M).$$

In other words,

$$h_{2m-k}(M, \omega) = \dim \mathrm{Im}\{L^{m-k} : H^k(M) \to H^{2m-k}(M)\}.$$

The following fact can be deduced from 4.3 using standard arguments from linear algebra, see [12].

Lemma 4.4. *Let ω_1 and ω_2 be two symplectic forms on a closed manifold M^{2m}. Suppose that, for $k = 1$ or $k = 2$, we have*

$$h_{2m-k}(M, \omega_1) \neq h_{2m-k}(M, \omega_2).$$

Then M is flexible.

Proposition 4.5. *Let G be a simply connected* 6*-dimensional nilpotent Lie group such that its Lie algebra $\mathfrak{g}$ has the basis $\{X_i\}_{i=1}^6$ and the following structure relations*:

$$[X_1, X_2] = -X_4, \qquad [X_1, X_4] = -X_5,$$
$$[X_1, X_5] = [X_2, X_3] = [X_2, X_4] = -X_6$$

(*all the other brackets $[X_i, X_j]$ are assumed to be zero*). *Then G admits a lattice Γ, and the corresponding compact nilmanifold $N := G/\Gamma$ admits two symplectic forms ω_1 and ω_2 such that*

$$\dim \mathrm{Im}\, L^2_{[\omega_1]} = 0, \qquad \dim \mathrm{Im}\, L^2_{[\omega_2]} = 2.$$

Proof. First, G has a lattice by 2.4(ii). Furthermore, by 2.4(iii), in the Chevalley–Eilenberg complex $(\Lambda^* \mathfrak{g}^*, d)$ we have

$$\begin{aligned} d\alpha_1 &= d\alpha_2 = d\alpha_3 = 0, \\ d\alpha_4 &= \alpha_1\alpha_2, \\ d\alpha_5 &= \alpha_1\alpha_4, \\ d\alpha_6 &= \alpha_1\alpha_5 + \alpha_2\alpha_3 + \alpha_2\alpha_4, \end{aligned}$$

where we write $\alpha_i\alpha_j$ instead of $\alpha_i \wedge \alpha_j$. One can check that the following elements represent closed homogeneous 2-forms on N:

$$\begin{aligned} \omega_1 &= \alpha_1\alpha_6 + \alpha_2\alpha_5 - \alpha_3\alpha_4, \\ \omega_2 &= \alpha_1\alpha_3 + \alpha_2\alpha_6 - \alpha_4\alpha_5. \end{aligned}$$

Since $[\omega_1^3] \neq 0 \neq [\omega_2^3]$, these homogeneous forms are symplectic. Indeed, by 2.4(iii) the cohomology classes $[\omega_0]$ and $[\omega_1]$ have homogeneous representatives whose third powers are non-zero. Then the same is valid for their pull-backs to *invariant* 2-forms on the Lie group G. But for invariant 2-forms this condition implies non-degeneracy. Since $G \to N$ is a covering, the homogeneous forms ω_1 and ω_2 on N are also non-degenerate. $\square$

Obviously, the $\mathbb{R}$-vector space $H^1(N)$ has the basis $\{[\alpha_1], [\alpha_2], [\alpha_3]\}$. One can check by direct calculation that

$$[\omega_1]^2[\alpha_i] = 0, \quad i = 1, 2, 3$$

and that

$$[\omega_2]^2[\alpha_1] = -2[\alpha_1\alpha_2\alpha_4\alpha_5\alpha_6], \qquad [\omega_2]^2[\alpha_2] = 0,$$
$$[\omega_2]^2[\alpha_3] = 2[\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6].$$

Finally, it is straightforward that the above cohomology classes span 2-dimensional subspace in $H^5(N)$.

Theorem 4.6. *There exists a flexible* 6*-dimensional nilmanifold.*

Proof. Consider the nilmanifold N as in 4.5. Because of 4.3 and 4.5, we conclude that

$$h_5(N, \omega_1) = 0 \neq 2 = h_5(N, \omega_2),$$

and the result follows from 4.4. □

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Addendum (November 2001)

During the preparation of the paper for the publication, we made some progress concerning flexibility. Namely, now we know that, for 6-dimensional closed manifolds, the numbers h_i for $i = 3, 4, 5$ can vary, and there are precisely ten 6-dimensional nilmanifolds with varying h_i [12,13].

We also made some progress concerning the tables from Theorem 3.1. Namely, in Theorem A.1 below we construct a manifold V with the properties from line 4 of Table 2.

Theorem A.1. *There exists a closed simply-connected symplectic manifold V,* $\dim V = 8$ *such that $b_3(V) = 1$ and all the Massey products in V are trivial.*

Proof. Let (M, ω) be a 4-dimensional symplectic manifold with $b_1(M) = 1$. The existence of such manifolds follows from results of Gompf [7]. Without loss of generality we can assume that the symplectic form on M is integral. We embed M in $\mathbb{CP}^5$ symplectically and denote by X the result of the blow up along this embedding. So, we have a commutative diagram

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\subset"] \arrow[d, "q"'] & X \arrow[d, "p"] \\ M \arrow[r, "\subset"] & \mathbb{CP}^5 \end{tikzcd}$$

where $q : \widetilde{M} \to M$ is a locally trivial bundle with the fiber $\mathbb{CP}^2$ and $p : X \setminus \widetilde{M} \to \mathbb{CP}^5 \setminus M$ is a diffeomorphism.

Let $\mathbb{CP}^1_1$ be a projective line in $\mathbb{CP}^5$ which does not meet M. Then the submanifold $p^{-1}(\mathbb{CP}^1_1)$ gives us a class $a \in H_2(X)$. Similarly, the inclusion $\mathbb{CP}^1_2 \subset \mathbb{CP}^2 \subset \widetilde{M} \subset X$ gives us a class $b \in H_2(X)$. It is well known that $\{a, b\}$ is a basis of $H_2(X)$ [21,29].

Let $\rho \in H^2(X)$ be the cohomology class of $p^*\omega_0$. It is clear that $\rho(a) \neq 0$ and $\rho(b) = 0$. It is well known that X possesses a symplectic form Ω whose cohomology class $[\Omega]$ is $\rho + \varepsilon\sigma$ for ε small enough. It is also easy to see that $b_3(X) = b_1(M) = 1$, see loc. cit.

Let ν be the normal bundle of the inclusion i: $\widetilde{M} \subset X$, and let $U \in H^*(T\nu)$ be the Thom class of ν. Consider the Browder–Novikov collapsing map $c : X \to T\nu$ and set $\sigma = c^*U$. It is easy to see that $\sigma(a) = 0$ and $\sigma(b) \neq 0$. Furthermore, for every $x \in H^*(X)$ we have

$$\sigma x = c^*\big(U(i^*x)\big). \tag{1}$$

Notice that in $H^*(X)$ we have $\rho^5 \neq 0 \neq \rho^2\sigma^2$. Indeed, $\rho^5 = p^*([\omega_0])^5$, and the restriction of $\rho^2\sigma^2$ on $\widetilde{M}$ is (up to a non-zero multiplicative constant) the top class $\xi q^*\omega \in H^8(\widetilde{M})$, where ξ restricts to a non-zero element of the fiber $\mathbb{CP}^2$.

Lemma A.2. *In X we have*

$$\rho^3\sigma = 0, \quad \rho^4 \cap [X] = k_1 a, \quad \rho^2\sigma^2 \cap X = k_2 b, \quad k_1, k_2 \neq 0.$$

Proof. The equality $\rho^3\sigma = 0$ follows from the equalities (1) and $\omega^3 = 0$.

Furthermore,

$$\rho \cap \big(\rho^4 \cap [X]\big) = \rho^5 \cap [X] \neq 0, \qquad \sigma \cap \big(\rho^4 \cap [X]\big) = \big(\rho^4\sigma\big) \cap [X] = 0$$

and hence $\rho^4 \cap [X] = k_1 a,\ k_1 \neq 0$. Finally,

$$\rho \cap \big(\rho^2\sigma^2 \cap [X]\big) = \big(\rho^3\sigma^2\big) \cap [X] = 0$$

and hence $\rho^2\sigma^2 \cap [X] = k_2 b,\ k_2 \in \mathbb{R}$. Thus, $k_2 b \neq 0$ since $\rho^2\sigma^2 \neq 0$. $\square$

By the routine arguments, we can assume that Ω is an integral form (by choosing a suitable ε). Because of the Donaldson Theorem [5, Theorem 1 and Proposition 39], there is a closed symplectic submanifold V of X of codimension 2 (i.e., $\dim V = 8$) such that the homology class $j_*[V]$ is dual to $N[\Omega]$ for N large enough; here $j : V \to X$ denotes the inclusion. In other words, $j_*[V] = (\lambda\rho + \mu\sigma) \cap [X]$ for some $\lambda, \mu \in \mathbb{R}$. Furthermore, $j_* : \pi_i(V) \to \pi_i(X)$ is an isomorphism for $i \leqslant 3$ and an epimorphism for $i = 4$. So, according to the Hurewicz–Whitehead Theorem, the homomorphism $j_* : H_i(V) \to H_i(X)$ is an isomorphism for $i \leqslant 3$ and an epimorphism for $i = 4$. In particular, $b_3(V) = 1$.

We set $u = j^*\rho$ and $v = j^*\sigma$.

Lemma A.3. *The $\mathbb{R}$-vector space $H^6(V)$ has dimension 2 and is generated by u^3 and u^2v.*

Proof. We have $H^6(V) = H_2(V) = H_2(X) = \mathbb{R}^2$. Furthermore, by Lemma 1,

$$\begin{aligned} j_*\big(u^3 \cap [V]\big) &= \rho^3 \cap \big(j_*[V]\big) = \rho^3 \cap (\lambda\rho + \mu\sigma) \cap [X] = \big(\lambda\rho^4 + \mu\rho^3\sigma\big) \cap [X] \\ &= \lambda\rho^4 \cap [X] = \lambda' a, \quad \lambda' \neq 0. \end{aligned}$$

Similarly,

$$\begin{aligned} j_*\big(u^2v \cap [V]\big) &= \rho^2\sigma \cap \big(j_*[V]\big) = \rho^2\sigma \cap (\lambda\rho + \mu\sigma) \cap [X] \\ &= \big(\lambda\rho^3\sigma + \mu\rho^2\sigma^2\big) \cap [X] = \mu\rho^2\sigma^2 \cap [X] = \mu' b, \quad \mu' \neq 0. \end{aligned}$$

Thus, u^3 and u^2v are linearly independent over $\mathbb{R}$. □

Now we prove that every Massey product $\langle\alpha, \beta, \gamma\rangle$ in $H^*(V)$ is trivial, i.e., it contains zero provided that it is defined. Notice that all the products u^2, uv and v^2 are non-zero because $j^* : H^4(X) \to H^4(V)$ is a monomorphism.

Case 1. $\langle\alpha, \beta, \gamma\rangle \subset H^5(V)$, i.e., $\alpha| = |\beta| = |\gamma| = 2$. Then the Massey product $\langle\alpha, \beta, \gamma\rangle$ is not defined, since $\alpha\beta \neq 0$.

Case 2. $\langle\alpha, \beta, \gamma\rangle \subset H^6(V)$. Clearly, if $\alpha \neq 0 \neq \gamma$ and $\langle\alpha, \beta, \gamma\rangle$ is defined then $|\beta| = 3$. Furthermore, if $\alpha = u$ or $\gamma = u$ then $H^6(V) = (\alpha, \gamma)$, and so $0 \in \langle\alpha, \beta, \gamma\rangle$. So, it remains to consider the case $\langle v, \beta, v\rangle$. But every Massey triple product of the form $\langle x, y, x\rangle$, $|x| = 2$, $|y| = 3$ contains zero. You can see it directly or use the equality

$$\langle x, y, x\rangle = -\langle x, y, x\rangle$$

from [14, Theorem 8].

Case 3. $\langle\alpha, \beta, \gamma\rangle \subset H^7(V) = 0$. Trivial.

Case 4. $\langle\alpha, \beta, \gamma\rangle \subset H^8(V) = \mathbb{R}$. Then there exist $z \in H^*(X)$ with $\alpha z \neq 0$, and so $(\alpha, \gamma) = H^8(V)$. Thus, $\langle\alpha, \beta, \gamma\rangle$ is trivial.

Finally, all the higher (i.e., quadruple, etc.) Massey products are trivial for the dimensional reasons. □

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